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The free interaction of the wall layers with the core of Couette–Poiseuille flow☆

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ABSTRACT

A method for the asymptotic description of one of the mechanisms by which fluctuations develop in plane Couette–Poiseuille flow at high Reynolds numbers is proposed. The class of wave perturbations of comparatively high amplitude which obey the linear Korteweg–de Vries equation despite the general non-linear multistage asymptotic structure of the flow field is indicated. The evolution of a localized perturbation and its conversion into a wave packet is considered.

The scenarios of the breakdown of Poiseuille laminar flow that have been investigated in most detail began with the linear amplification of Tollmien-Schlichting waves. The first theoretical results on the stability of Poiseuille flow were published by Heisenberg,[1] who established, by asymptotic methods, the position of one of the branches of the neutral curve. Subsequent theoretical and experimental investigations, refining Heisenberg's conclusions, enabled, in particular, the critical Reynolds number to be determined. The applicability of the linear theory of hydrodynamic stability for plane Poiseuille flow was finally confirmed.[2,3]

A quite different situation arises in attempts to reconcile the theoretical and experimental approaches when investigating the role of different kinds of perturbations for Couette flow. Disagreements on the reasons for the occurrence of instability in plane Couette flow arose most clearly in relation to the theoretically predicted absence of a neutral curve of linear perturbations. It was proved,[4,5] that Couette flow remains stable in the linear approximation for all Reynolds numbers. Nevertheless, the experimentally observed destabilization of plane Couette flow[6,7] makes it necessary to make a considerable correction to the theoretical models.

Below we construct one of the modifications of perturbation theory for a combination of Couette and Poiseuille flows. An interesting fact (which is not completely obvious in advance) is the applicability of multistage asymptotic constructions, introduced in the theory of free interactions of the boundary layer,[8–10] to describe the loss of stability of viscous flows.[11,12] The technique of asymptotic expansions[8–12] enabled excitations of pulsation fields in Couette–Poiseuille flow,[13,14] which in principle does not occur in Poiseuille flow, to be explained. An asymptotic analysis confirms the existence of four types of neutral (or close to neutral) natural oscillations, which differ in having a different arrangement of the critical and wall sublayers.

In this paper from the very beginning we take as the initial equations the Navier–Stokes equations, on the assumption that the Reynolds numbers are high. Unlike the previous analysis[14] we consider a non-linear version of the asymptotic theory, which leads to the Korteweg–de Vries equation for describing the wave pattern of the perturbed flow. Note that the stability of Couette flow to non-linear perturbations is the object of discussions in Refs. 6, 7 and 13 from the point of view of the probable explanation of the transition to a turbulent state.

1. Formulation of the boundary-value problem

The unperturbed Couette–Poiseuille flow in a channel with moving walls (Fig. 1) when there is a constant pressure gradient $\partial p^*/\partial x^* = -|\partial p_0^*/\partial x^*|$ is given by the functions

$$u^* = u_0^*(y^*) = \frac{1}{2\nu^*\rho^*}\left|\frac{dp_0^*}{dx^*}\right|[b^{*2} - y^{*2}] + u_w^*\frac{y^*}{b^*}, \quad \upsilon^* = 0 \tag{1.1}$$

which satisfy the Navier–Stokes equations

$$\frac{\partial \mathbf{u}^*}{\partial t^*} + (\mathbf{u}^* \cdot \nabla_*)\mathbf{u}^* = -\frac{1}{\rho^*}\nabla_* p^* + \nu^*\nabla_*^2\mathbf{u}^*, \ \ \nabla_* \cdot \mathbf{u}^* = 0,$$

$$\mathbf{u}^* = \{u^*, \upsilon^*\}, \ \ \nabla_* = \left\{\frac{\partial}{\partial x^*}, \frac{\partial}{\partial y^*}\right\}$$

and the boundary conditions

$$y^* = b^*: u^* = u_w^*, \quad \upsilon^* = 0; \quad y^* = -b^*: u^* = -u_w^*, \quad \upsilon^* = 0$$

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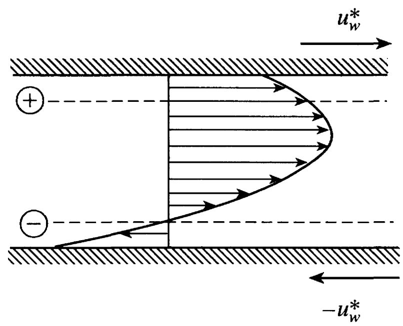

Fig. 1.

Solving (1.1) we determine the average value of the velocity of motion of an incompressible fluid

$$U_m^* = \frac{1}{2b^*}\int_{-b^*}^{b^*} u_0^* dy^* = \frac{b^{*2}}{3\nu^*\rho^*}\left|\frac{\partial p_0^*}{\partial x^*}\right|$$

which is uniquely related to the pressure in the channel

$$p^* = p_0^*(x^*) = -3\nu^*\rho^* U_m^* x^*/b^{*2}$$

It is natural to choose the half-width of the channel b^*, the mean velocity U_m^* over the channel cross-section and the density ρ^* of the incompressible fluid in order to normalize the coordinates of space $\{x^*, y^*\} = \{b^* x, b^* y\}$, the time $t^* = b^* U_m^{*-1}$, the components of the velocity vector $\mathbf{u}^* = \{u^*, v^*\} = \{U_m^* u, U_m^* v\} = U_m^* \mathbf{u}$ and the pressure $p^* = \rho * U_m^{*2} p$. In the new variables, the initial stationary solution of the Navier–Stokes equations

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} = -\nabla p + \frac{1}{\mathrm{Re}}\nabla^2\mathbf{u}, \quad \nabla\cdot\mathbf{u} = 0, \quad \nabla = \left\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right\} \tag{1.2}$$

which take the form

$$u = u_0(y) = \frac{3}{2}(1-y^2) + u_w y, \quad v = v_0 = 0, \quad p = p_0(x) = -\frac{3}{\mathrm{Re}}x \tag{1.3}$$

and which satisfy the boundary conditions

$$y = 1: u = u_w, \quad v = 0; \quad y = -1: u = -u_w, \quad v = 0 \tag{1.4}$$

is determined by the velocities of the channel walls $\pm u_W = \pm u_W^* U_m^{*-1}$ and Reynolds number $\mathrm{Re} = U_m^* b^* \nu^{*-1}$, where ν^* is the kinematic viscosity.

2. An asymptotic description of the perturbed core of Couette–Poiseuille flow and the wall layers

We will put $\mathrm{Re}\to\infty$ and consider the special class of perturbations

$$u = u_0(Y) + \mathrm{Re}^{-2/7}u_1(T, X, Y) + \ldots, \quad v = \mathrm{Re}^{-3/7}v_1(T, X, Y) + \ldots$$
$$p = p_0(X) + \mathrm{Re}^{-4/7}p_1(T, X, Y) + \ldots \tag{2.1}$$

of the exact solution (1.3). The variables

$$T = \mathrm{Re}^{-3/7}t, \quad X = \mathrm{Re}^{-1/7}x, \quad Y = y \tag{2.2}$$

are the arguments of the perturbing functions in (2.1), and hence not only the amplitude but also the space-time characteristics of the fluctuations are specified in terms of certain powers of the Reynolds number. The form of the asymptotic sequences (2.1) reflects the singular nature of the small parameter Re^{-1} occurring in the Navier–Stokes Eq. (1.2), from which we have the closed system of equations in the perturbations

$$u_0\frac{\partial u_1}{\partial X} + v_1\frac{du_0}{\partial Y} = 0, \quad u_0\frac{\partial v_1}{\partial X} = -\frac{\partial p_1}{\partial Y}, \quad \frac{\partial u_1}{\partial X} + \frac{\partial v_1}{\partial Y} = 0 \tag{2.3}$$

The solutions of Eq. (2.3) are defined, apart from two arbitrary functions $A_1(T, X)$ and $P_1(T, X)$

$$u_1 = A_1(T, X)\frac{du_0(Y)}{dY}, \quad v_1 = -u_0(Y)\frac{\partial A_1(T, X)}{\partial X}$$
$$p_1 = P_1(T, X) + \frac{\partial^2 A_1(T, X)}{\partial X^2}\int_0^Y u_0^2(Y')dY' \tag{2.4}$$

Representation (2.1) is unsuitable in the neighbourhoods of the channel walls. In fact, on the boundaries $Y = \pm 1$ of the region occupied by the fluid, the longitudinal velocity

$$u(t, x, \pm 1) = u_0(\pm 1) + \mathrm{Re}^{-2/7}u_1(T, X, \pm 1)$$

is not the same as the velocities of the moving walls $\pm u_W$, since $u_0(\pm 1) = \pm u_W$, by virtue of the first equation of (1.3) and

$$u_1(T, X, \pm 1) = (\mp 3 + u_w)A_1(T, X)$$

in accordance with the first equation of (2.4)

Moreover, expansions (2.1) for the core of the flow $Y = O(1)$ in the channel only hold when the conditions of asymptotic matching hold in the expansions in narrow subregions close to the walls, where the equations describing the motion have a viscous nature, unlike Eq. (2.3). We will show that this matching is possible on the assumption that

$$u_w = \mathrm{Re}^{-2/7}\bar{u}_w, \quad \bar{u}_w = O(1) \tag{2.5}$$

In the region of the upper wall $1 - Y = O(\mathrm{Re}^{-2/7})$ and the lower wall $1 + Y = O(\mathrm{Re}^{-2/7})$ of the channel, instead of (2.1) we introduce the new expansions

$$u = \mathrm{Re}^{-2/7}u_\pm(T, X, Y_\pm) + \ldots, \quad v = \mp\mathrm{Re}^{-5/7}v_\pm(T, X, Y_\pm) + \ldots$$
$$p = \mathrm{Re}^{-4/7}p_\pm(T, X, Y_\pm) + \ldots \tag{2.6}$$

The hydrodynamic functions here are indicated by a plus subscript for the upper wall sublayer and a minus subscript for the lower sublayer (Fig. 1), while the new (extended) vertical coordinates $Y_\pm = O(1)$ are defined as follows:

$$Y_+ = \mathrm{Re}^{2/7}(1 - Y), \quad Y_- = \mathrm{Re}^{2/7}(1 + Y) \tag{2.7}$$

We introduce representations (2.6) into the Navier–Stokes Eq. (1.2). We then obtain the following two systems of equations for the functions with plus and minus subscripts

$$\frac{\partial u_\pm}{\partial T} + u_\pm\frac{\partial u_\pm}{\partial X} + v_\pm\frac{\partial u_\pm}{\partial Y_\pm} = -\frac{\partial p_\pm}{\partial X} + \frac{\partial^2 u_\pm}{\partial Y_\pm^2}, \quad \frac{\partial p_\pm}{\partial Y_\pm} = 0, \quad \frac{\partial u_\pm}{\partial X} + \frac{\partial v_\pm}{\partial Y_\pm} = 0 \tag{2.8}$$

The inner limit $(Y_\pm \to +\infty)$ of expansions (2.1), written in terms of the variables $Y_\pm$ serve as the outer boundary conditions $(Y \to \pm 1)$ (for Eq. (2.8)). From expansions (2.1), taking the solutions (2.4) into account, we have for $Y \to \pm 1$

$$u = 3(1 \mp Y) + \mathrm{Re}^{-2/7}[\pm\bar{u}_w \mp 3A_1(T, X)] + \ldots$$
$$p = \mathrm{Re}^{-4/7}\left[P_1(T, X) \pm \frac{6}{5}\frac{\partial^2 A_1(T, X)}{\partial X^2}\right] + \ldots \tag{2.9}$$

Changing here to variables (2.7), it can be verified that when the conditions

$$Y_\pm \to +\infty:\ u_\pm \to 3Y_\pm \pm \bar{u}_w \mp 3A_1(T,X)$$
$$p_\pm(T,X) = P_1(T,X) \pm \frac{6}{5}\frac{\partial^2 A_1(T,X)}{\partial X^2} \tag{2.10}$$

are satisfied, functions (2.1) and (2.6) change into one another, i.e. matching of the asymptotic expansions is obtained in the core of the perturbed Couette–Poiseuille flow and in the two wall subregions.

The addition of the no-slip boundary conditions for the fluid particles at the channel walls

$$Y_\pm = 0:\ u_\pm = \pm\bar{u}_w, \quad v_\pm = 0 \tag{2.11}$$

closes the problem.

In problem (2.8), (2.10) and (2.11) the two systems of equations (for the functions with plus and minus subscripts) are not independent, since the boundary conditions contain the two unknown functions $P_1(T, X)$ and $A_1(T, X)$, common to both systems.

Although Eq. (2.8) are identical in form with Prandtl's equations, unlike the classical Prandtl theory the pressure gradient pertains to a number of the required quantities. The description of the evolution of the perturbations must therefore be reduced to the problem of the free interaction of the core of the Couette–Poiseuille flow with two wall subregions. The latter, in accordance with the terminology used in Refs. 8–10, are boundary layers with self-induced pressure.

Note that the set of solutions of boundary-value problem (2.8), (2.10), (2.11) is not empty, as follows from the previously investigated[14] linear approximation – Tollmien-Schlichting waves (assuming small amplitudes). Below we consider, in a certain sense, the opposite limiting case of large amplitudes of the pulsation fields.

3. Locally-non-viscous non-linear perturbations and the Korteweg–de Vries equation

We will introduce the parameter χ and carry out a transformation of the dependent and independent variables of the following form

$$u_\pm = \chi^2\hat{u}_\pm, \quad v_\pm = \chi^5\hat{v}_\pm, \quad p_\pm = \chi^4\hat{p}_\pm, \quad P_1 = \chi^4\hat{P}_1$$
$$T = \chi^{-3}\hat{T}, \quad X = \chi^{-1}\hat{X}, \quad Y_\pm = \chi^2\hat{Y}_\pm, \quad A_1 = \chi^2\hat{A}_1, \quad \bar{u}_w = \chi^2\hat{u}_w \tag{3.1}$$

Substituting expressions (3.1) into relations (2.8), (2.10) and (2.11) we obtain

$$\frac{\partial\hat{u}_\pm}{\partial\hat{T}} + \hat{u}_\pm\frac{\partial\hat{u}_\pm}{\partial\hat{X}} + \hat{v}_\pm\frac{\partial\hat{u}_\pm}{\partial\hat{Y}_\pm} = -\frac{\partial\hat{p}_\pm}{\partial\hat{X}} + \chi^{-7}\frac{\partial^2\hat{u}_\pm}{\partial\hat{Y}_\pm^2}, \quad \frac{\partial\hat{p}_\pm}{\partial\hat{Y}_\pm} = 0, \quad \frac{\partial\hat{u}_\pm}{\partial\hat{X}} + \frac{\partial\hat{v}_\pm}{\partial\hat{Y}_\pm} = 0$$
$$\hat{Y}_\pm = 0:\ \hat{u}_\pm = \pm\hat{u}_w, \quad \hat{v}_\pm = 0; \quad \hat{Y}_\pm \to +\infty:\ \hat{u}_\pm \to 3\hat{Y}_\pm \pm \hat{u}_w \mp 3\hat{A}_1(\hat{T},\hat{X})$$
$$\hat{p}_\pm(\hat{T},\hat{X}) = \hat{P}_1(\hat{T},\hat{X}) \pm \frac{6}{5}\frac{\partial^2\hat{A}_1(\hat{T},\hat{X})}{\partial\hat{X}^2} \tag{3.2}$$

The last equality in system (3.2), among the set of solutions of the equation $\partial\hat{p}_\pm/\partial\hat{Y}_\pm = 0$ distinguishes those the form of which agrees with the matching procedure on the outer boundaries of the wall sublayers.

Conversion (3.1) therefore leaves all the relations (2.8), (2.10) and (2.11) unchanged, with the exception of the first equation of (2.8) in which, when $\chi \to +\infty$, a small parameter occurs for the leading derivative (corresponding to viscous shear stresses). Dropping the term of the order of χ^{-7} in this equation and assuming everywhere henceforth that $\chi \to +\infty$, we obtain the following system of non-viscous equations

$$\frac{\partial\hat{u}_\pm}{\partial\hat{T}} + \hat{u}_\pm\frac{\partial\hat{u}_\pm}{\partial\hat{X}} + \hat{v}_\pm\frac{\partial\hat{u}_\pm}{\partial\hat{Y}_\pm} = -\frac{\partial\hat{p}_\pm}{\partial\hat{X}}, \quad \frac{\partial\hat{p}_\pm}{\partial\hat{Y}_\pm} = 0, \quad \frac{\partial\hat{u}_\pm}{\partial\hat{X}} + \frac{\partial\hat{v}_\pm}{\partial\hat{Y}_\pm} = 0 \tag{3.3}$$

It is easy to check that the expressions

$$\hat{u}_\pm = 3\hat{Y}_\pm \pm \hat{u}_w \mp 3\hat{A}_1,$$
$$\hat{v}_\pm = \pm 3\hat{Y}_\pm\frac{\partial\hat{A}_1}{\partial\hat{X}} \pm \frac{\partial\hat{A}_1}{\partial\hat{T}} - 3\hat{A}_1\frac{\partial\hat{A}_1}{\partial\hat{X}} + \hat{u}_w\frac{\partial\hat{A}_1}{\partial\hat{X}} - \frac{1}{3}\frac{\partial\hat{p}_\pm}{\partial\hat{X}}$$
$$\hat{p}_\pm = \hat{P}_1 \pm \frac{6}{5}\frac{\partial^2\hat{A}_1}{\partial\hat{X}^2} \tag{3.4}$$

satisfy both the equations of the boundary-value problem (3.2) (for any χ), and Eq. (3.3).

Solution (3.4), moreover, satisfies the limit conditions from (3.2) as $\hat{Y}_\pm \to +\infty$, and also the relations which connect the functions $\hat{p}_\pm$, $\hat{P}_1$ and $\hat{A}_1$ in (3,2). As regards the boundary conditions on the solid surface around which the flow occurs, in accordance with solution (3.4), to satisfy the impermeability conditions

$$\hat{Y}_\pm = 0:\ \hat{v}_\pm = 0 \tag{3.5}$$

it is necessary and sufficient to satisfy the following two equalities simultaneously (choosing only the upper and only the lower signs)

$$\frac{\partial\hat{A}_1}{\partial\hat{T}} \mp 3\hat{A}_1\frac{\partial\hat{A}_1}{\partial\hat{X}} \pm \hat{u}_w\frac{\partial\hat{A}_1}{\partial\hat{X}} = \pm\frac{1}{3}\frac{\partial\hat{P}_1}{\partial\hat{X}} + \frac{2}{5}\frac{\partial^3\hat{A}_1}{\partial\hat{X}^3} \tag{3.6}$$

The change from system (2.8) to system (3.3) leads to a loss of the no-slip boundary condition

$$\hat{Y}_\pm = 0:\ \hat{u}_\pm = \pm\hat{u}_w \tag{3.7}$$

due to the reduction in the order of the system. If we consider expressions (3.4) as the asymptotic form of the solution of system (2.8) when $\chi \to +\infty$, then it is suitable everywhere, with the exception of thin sublayers (on the scale of the variables $\hat{Y}_\pm$), adjacent to the channel walls $\hat{Y}_\pm = 0$. Hence, the wall subregions described by Eq. (2.8) when $\chi \to +\infty$ are themselves split into non-viscous zones $\hat{Y}_\pm = O(1)$ and viscous wall sublayers $\hat{Y}_\pm \to 0$. Since, in these wall sublayers, the derivatives with respect to the vertical coordinate $\hat{Y}_\pm$ are large, then, in the first equation of system (3.3), the term with the small parameter χ^{-7} must be retained as the coefficient. The existence of wall sublayers enables us to satisfy the impermeability condition (3.5) and the no-slip condition (3.7).

Summation of the two Eq. (3.6) eliminates the function $\hat{P}_1$ and leads to a linearized Korteweg–de Vries equation

$$\frac{\partial\hat{A}_1}{\partial\hat{T}} = \frac{2}{5}\frac{\partial^3\hat{A}_1}{\partial\hat{X}^3} \tag{3.8}$$

Subtracting Eq. (3.6), we obtain, after integration,

$$\hat{P}_1 = 3\hat{u}_w\hat{A}_1 - \frac{9}{2}\hat{A}_1^2 \tag{3.9}$$

Although the equation for determining $\hat{A}_1$ turned out to be linear, the problem as a whole, as relation (3.9) shows, remains non-linear.

The search for wave solutions of Eq. (3.8) in the form

$$\hat{A}_1 = \epsilon\exp[ik(\hat{X} - c\hat{T})]$$

establishes the dispersion relation

$$c = 2k^2/5$$

which represents the principal part of the previously obtained[14] spectral equality for the first mode of natural oscillations of a viscous fluid.

By renormalizing the variables (3.1) and then taking the limit as $\chi \to +\infty$, we can formulate the non-viscous problem within the framework of the initial asymptotic Eq. (3.8) of the motion of a viscous fluid. However, the new variables (3.1) enable us to propose a different version of the construction of the theory of slightly non-linear perturbations of Couette–Poiseuille flow at the stage of the asymptotic expansions of the solution of the initial Navier–Stokes equation. We will introduce the parameter

$$\alpha = \mathrm{Re}^{-2/7}\chi^2 \tag{3.10}$$

which we will assume to be small (although simultaneously $\mathrm{Re} \to \infty$, $\chi \to +\infty$, but $\alpha \to 0$). The new version of the expansions in asymptotic series can be obtained by interchanging the dependent and independent variables in relations (2.1) and (2.6) using formulae (3.1) and eliminating the parameter χ in accordance with equality (3.10): $\chi = \alpha^{1/2}\mathrm{Re}^{1/7}$.

Using the above discussions as our basis, instead of expansions (2.6) we can consider the asymptotic series

$$u = \alpha\hat{u}_\pm(\hat{T}, \hat{X}, \hat{Y}_\pm) + \ldots, \quad v = \alpha^{5/2}\hat{v}_\pm(\hat{T}, \hat{X}, \hat{Y}_\pm) + \ldots,$$
$$p = \alpha^2\hat{p}_\pm(\hat{T}, \hat{X}, \hat{Y}_\pm) + \ldots \tag{3.11}$$

We have as the arguments of the functions on the right-hand sides

$$\hat{T} = \alpha^{3/2}t, \quad \hat{X} = \alpha^{1/2}x, \quad \hat{Y}_\pm = (1 \mp Y)\alpha^{-1}$$

In the basic thickness of the channel, expansion (2.1) becomes

$$u = u_0(Y) + \alpha\hat{u}_1(\hat{T}, \hat{X}, Y) + \ldots, \quad v = \alpha^{3/2}\hat{v}_1(\hat{T}, \hat{X}, Y) + \ldots,$$
$$p = p_0(X) + \alpha^2\hat{p}_1(\hat{T}, \hat{X}, Y) + (3.12) \tag{3.12}$$

Introducing the representations (3.11) and (3.12) into the system of Navier–Stokes equations, it can be shown that the result of matching the wall layers with the core of the Couette–Poiseuille flow leads to boundary-value problem (3.2), where the term with the second derivative $\partial^2\hat{u}_\pm/\partial\hat{Y}_\pm^2$ in the first equation of system (3.2) must be dropped, since the parameter $\chi^{-7} = \alpha^{-7/2}\mathrm{Re}^{-1}$, which is small by virtue of relation (3.10), occurs in it as a coefficient. Expressions similar to (2.4) then remain in force, and the boundary-value problem (3.2) is reduced to solving the Korteweg–de Vries equation for a single unknown function $\hat{A}_1(\hat{T}, \hat{X})$.

4. Transformation of the initial perturbations into a wave packet

After making the change of variables

$$t = 2\hat{T}/5, \quad x = \hat{X}, \quad A(t, x) = \hat{A}_1(\hat{T}, \hat{X})$$

the linearized Korteweg–de Vries Eq. (3.8) acquires the canonical form

$$\frac{\partial A}{\partial t} = \frac{\partial^3 A}{\partial x^3} \tag{4.1}$$

Suppose we set up the Cauchy problem for Eq. (4.1) with initial condition

$$A|_{t=0} = A_0(x) \tag{4.2}$$

The symbols $F[\varphi]$ and $F^{-1}[\psi]$ will henceforth denote the direct and inverse Fourier transformations with respect to the variable x. The classical Fourier transformations are defined as

$$F[\varphi] = \int_{-\infty}^{\infty} \varphi(t, x)\exp(ikx)dx, \quad F^{-1}[\psi] = \frac{1}{2\pi}\int_{-\infty}^{\infty} \psi(t, k)\exp(-ikx)dk \tag{4.3}$$

We will use the concept of a generalized solution[15] as one of the possible forms of describing the properties of the solution of the Cauchy problem. We will denote by $\mathcal{E}(t, x)$ the fundamental solution of the Korteweg–de Vries operator, i.e. the generalized solution of the equation

$$\frac{\partial \mathcal{E}}{\partial t} - \frac{\partial^3 \mathcal{E}}{\partial x^3} = \delta(t, x) \tag{4.4}$$

where $\delta(t, x)$ is the Dirac delta function. The space of generalized functions $\mathcal{D}'$ consists of linear continuous functionals, denoted by $f(t, x)$, in the space of so-called fundamental functions $\varphi(t, x)$ from the space $\mathcal{D}$. The latter is a linear set of finite infinitely differentiable functions. The result of the action of the functional $f \in \mathcal{D}'$ on the fundamental function $\varphi \in \mathcal{D}$ is written as (f, φ). The generalized Fourier transformation $F[f(t, x)](t, \xi)$ in the space $\mathcal{D}'$ is defined by the equation

$$(F[f], \varphi) = (f, F[\varphi]) \tag{4.5}$$

for any fundamental function $\varphi \in \mathcal{D}$. On the right-hand side of Eq. (4.5) we have the classical Fourier transformation (4.3), and on the left-hand side we have the generalized Fourier transformation.

We will apply the generalized Fourier transformation with respect to the variable x to Eq. (4.4). This leads to the ordinary differential equation

$$\frac{\partial F[\mathcal{E}]}{\partial t} - (-i\xi)^3 F[\mathcal{E}] = \delta(t)$$

the solution of which in the space of generalized functions $\mathcal{D}'$ can be expressed in terms of the Heaviside theta function $\theta(t)$

$$F[\mathcal{E}] = \theta(t)\exp(i\xi^3 t) \tag{4.6}$$

The inverse Fourier transformation $F^{-1}[g(t, \xi)]$ with respect to the variable x in the space of generalized functions $\mathcal{D}'$ is expressed in terms of the direct Fourier transformation $F[f(t, x)]$ as follows:

$$F^{-1}[g(t, \xi)] = \frac{1}{2\pi}F[g(t, -\xi)] \tag{4.7}$$

Calculating the inverse Fourier transformation of the generalized function (4.6) using formula (4.7), we obtain

$$\mathcal{E}(t, x) = \frac{\theta(t)}{2\pi}\int_{-\infty}^{\infty} \exp(i\xi^3 t - i\xi x)d\xi \tag{4.8}$$

The classical solution of the Cauchy problem (4.1), (4.2) is contained among the generalized solutions of the problem

$$\frac{\partial A}{\partial t} - \frac{\partial^3 A}{\partial x^3} = A_0(x) \bullet \delta(t) \tag{4.9}$$

where the large dot denotes the direct product of generalized functions. As is well known,[15] the solution of generalized problem (4.9) is given by the convolution of the fundamental solution (4.8) and the generalized function on the right-hand side of Eq. (4.9)

$$A(t, x) = \mathcal{E}(t, x) * [A_0(x) \bullet \delta(t)] = \int_{-\infty}^{\infty} A_0(\xi)\mathcal{E}(t, x - \xi)d\xi \tag{4.10}$$

Fubini's theorem on the change in the order of integration, as it applies to integral (4.10), gives

$$A(t,x) = \int_{-\infty}^{\infty} A_0(\xi)d\xi \int_{-\infty}^{\infty} \frac{\theta(t)}{2\pi}\exp(ik^3t - ik(x-\xi))dk =$$

$$= \frac{\theta(t)}{2\pi}\int_{-\infty}^{\infty}\exp(ik^3t - ikx)dk\int_{-\infty}^{\infty}A_0(\xi)\exp(ik\xi)d\xi \quad (4.11)$$

The generalized solution of problem (4.9) in the upper half-plane $t > 0$ is also the classical solution of the Cauchy problem. In accordance with relation (4.11), the classical solution of problem (4.1), (4.2) for $t > 0$ is given by the formula

$$A(t,x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} A_0^*(k)\exp(\Phi(k))dk, \quad \Phi(k) = ik^3t - ikx \quad (4.12)$$

where $A_0^*(k)$ is the Fourier transform of the initial function $A_0(x)$ from condition (4.2)

$$A_0^*(k) = \int_{-\infty}^{\infty} A_0(x)\exp(ikx)dx \quad (4.13)$$

An asymptotic estimate of the integral (4.12) using the method of steepest descent as $t \to +\infty$ indicates the formation of a wave packet, propagating with a certain group velocity. We obtain the structure of the wave packet by the method of steepest descent.

5. Construction of the oscillation pattern of the perturbations by the method of steepest descent

We will consider the factor of the exponential function $\Phi(k)$ in solution (4.12) as an analytic function of the complex variable k. The stationary point (the point of steepest descent) is found from the condition $d\Phi/dk = 0$, i.e.

$$k = k_0 = \sqrt{x/(3t)} \quad (5.1)$$

We will assume $x/t = O(1)$, $t \to +\infty$. Using the fact that $A_0^*(-k) = \overline{A_0^*(k)}$ for real $A_0(x)$, where the bar denotes complex-conjugate quantities. The solution (4.12) can then be rewritten in the form

$$A(t,x) = \frac{1}{\pi}\mathrm{Real}\Pi, \quad \Pi = \int_0^{\infty} A_0^*(k)\exp(\Phi(k))dk \quad (5.2)$$

In the complex plane k, we will introduce the bundle of straight lines

$$k = k_0 + \varepsilon\exp(i\gamma) \quad (5.3)$$

passing through $k = k_0$. The straight line from the set (5.3) is distinguished by the parameter γ and the position of the point on the straight line is given by the distance ε from the centre $k = k_0$ of the bundle.

The function $\Phi(k)$ at points of each straight line of the family (5.3) has complex values

$$\Phi(k(\varepsilon)) = -2itk_0^3 + 3i\varepsilon^2tk_0\cos 2\gamma - 3\varepsilon^2tk_0\sin 2\gamma + it\varepsilon^3\exp(3i\gamma) \quad (5.4)$$

In accordance with the method of steepest descent we will deform the contour of integration in the complex plane k in such a way that it passes through the point of descent $k = k_0$ (a saddle point) in the direction of the quickest decrease in the modulus of the exponential function in expression (5.2). It can be seen from equality (5.4) that, in a small neighbourhood $\varepsilon \to 0$ of the saddle point $k = k_0$ of the analytic function $\Phi = \Phi_r + i\Phi_i$, its real part Φ_r, for each ε, takes its negative minimum value on the straight line $\gamma = \pi/4$. Hence, any section of this straight line containing the point $\varepsilon = 0$ is also a contour of steepest descent.

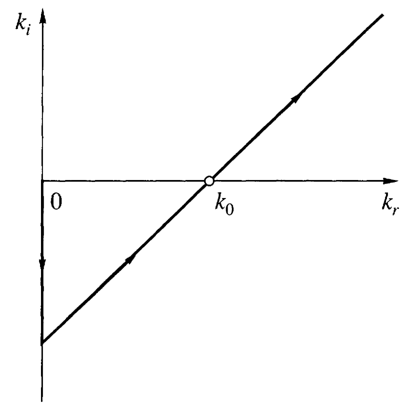

Fig. 2.

In integral (5.2), instead of the bundle (0, +∞) we will choose the integration path $C = C_1 \cup C_2$, consisting of the section C_1 on the imaginary axis and the bundle C_2, which intersects the real axis at the point k_0 in the direction of most rapid descent (Fig. 2), i.e.

$$C_1 = \{k = -is,\ 0 \le s \le k_0\},$$

$$C_2 = \left\{k = k_0 + \varepsilon\exp\left(\frac{i\pi}{4}\right),\ -\sqrt{2}k_0 \le \varepsilon \le +\infty\right\}$$

Integral (5.2) can be split into two integrals

$$\Pi = \left[\int_{C_1} + \int_{C_2}\right]A_0^*(k)\exp(ik^3t - ikx)dk = \Pi_1 + \Pi_2$$

$$\Pi_1 = -i\int_0^{k_0} A_0^*(-is)\exp(-s^3t - sx)ds$$

$$\Pi_2 = \int_{-\sqrt{2}k_0}^{+\infty} A_0^*\left(k_0 + \varepsilon\exp\left(\frac{i\pi}{4}\right)\right)\exp\left(\Phi(k(\varepsilon)) + \frac{i\pi}{4}\right)d\varepsilon$$

We will assume that the function $A_0(x)$ from condition (4.2) is absolutely integrable, for example, finite. We then have for its Fourier transform (4.13)

$$|A_0^*(k)| \le \mathcal{M}, \quad \mathcal{M} > 0$$

which leads to the inequalities

$$|\Pi_1| \le \int_0^{k_0}|A_0^*(-is)|\exp(-s^3t - 3sk_0^2t)ds \le \mathcal{M}\int_0^{k_0}\exp(-3sk_0^2t)ds \le \frac{\mathcal{M}}{3k_0^2t}$$

Consequently, $\Pi_1 \to 0$ as $t \to +\infty$ with a rate of decrease of $O(t^{-1})$.

For an asymptotic estimate of Π_2 we will split the semi-infinite path of integration $\mathcal{L} = [-\sqrt{2}k_0, +\infty]$ into three parts

$$\mathcal{L} = \mathcal{L}_- \cup \mathcal{L}_0 \cup \mathcal{L}_+, \quad \mathcal{L}_- = [-\sqrt{2}k_0, -\delta],$$

$$\mathcal{L}_0 = [-\delta, +\delta], \quad \mathcal{L}_+ = [+\delta, +\infty)$$

where we will assume the quantity $\delta > 0$ to be small but fixed ($\delta < \sqrt{2}k_0$). The function $\Phi(k(\varepsilon))$ in the index of the exponential function in the integral Π_2, according to expression (5.4), has the real part

$$\Phi_r(k(\varepsilon)) = -3tk_0\varepsilon^2 - \frac{t\varepsilon^3}{\sqrt{2}}$$

which decreases monotonically on both sides from the point $\varepsilon = 0$ of its maximum $\Phi_r(k(0)) = 0$, when ε runs through the whole set $\mathcal{L}$. The maximum of $\Phi_r(k(0))$ in the subset $\mathcal{L}_- \cup \mathcal{L}_+$ is reached at the point $\varepsilon = -\delta$ at the right end of the section $\mathcal{L}_-$. Hence, the following estimate holds

$$\varepsilon \in \mathcal{L}_- \cup \mathcal{L}_+: \left|\exp\left(\Phi(k(\varepsilon)) + \frac{i\pi}{4}\right)\right| \le \exp\left(-3tk_0\delta^2\left(1 - \frac{\delta}{3\sqrt{2}k_0}\right)\right) \tag{5.5}$$

It shows that when $t \to +\infty$, apart from exponentially small terms, it is sufficient to confine ourselves to integrating over $\mathcal{L}_0$, namely,

$$\Pi_2 = A_0^*(k_0)\exp\left(i\left(-2tk_0^3 + \frac{\pi}{4}\right)\right)\int_{-\delta\sqrt{3tk_0}}^{\delta\sqrt{3tk_0}} \exp(-\mu^2)[1 + O(\delta)]\frac{d\mu}{\sqrt{3tk_0}} + O(\exp(-3tk_0\delta^2))\,(5.6) \tag{5.6}$$

Formula (5.6) is obtained using expression (5.4) for $\Phi(k(\varepsilon))$ and making the substitution

$$\mu = \varepsilon\sqrt{3tk_0\left(1 + \frac{1+i}{3\sqrt{2}k_0}\varepsilon\right)}$$

We will first put $t \to +\infty$ for fixed δ. Then the integral in Eq. (5.6) changes into an improper Poisson integral equal to $\sqrt{\pi}$. Taking the limit as $\delta \to 0$, we finally obtain

$$\Pi_2 = \sqrt{\frac{\pi}{3tk_0}}A_0^*(k_0)\exp\left(i\left(-2tk_0^3 + \frac{\pi}{4}\right)\right)$$

Hence, the asymptotic method of steepest descent when applied to integral (4.12) as $t \to +\infty$ enables us to conclude that the initial perturbation develops into a wave packet

$$A(t, x) = \mathrm{Real}\left[\sqrt{\frac{\pi}{3tk_0}}A_0^*(k_0)\exp\left(i\left(-\frac{2}{3\sqrt{3}}x^{3/2}t^{-1/2} + \frac{\pi}{4}\right)\right)\right] \tag{5.7}$$

The group velocity of the wave packet is k_0^2, where $k_0 = x^{1/2}(3t)^{-1/2}$. Expression (5.7) holds for $xt^{-1} = O(1)$, $t \to +\infty$ and describes the occurrence of high-frequency oscillations as time passes and as one moves from the centre of the packet to its peripheral parts.

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